SMC IMONST Workshop 2024
(Day 1: Combinatorics)

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Contents

Before You Begin...

This handout was written specifically for a two-day workshop to prepare members of Sunway Mathematics Club for IMONST1 2024.

As of 2023, each IMONST1 test lasts for 2 1/2 hours and consists of 30 questions: first 10 questions are worth 1 point, next 10 questions are worth 3, and the last 10 questions are worth 6. The maximum score attainable is 100 points.

A typical IMONST1 test may cover the following topics as mentioned on their official website:

- **Number Theory:** Properties of Integers, Primes, Divisibility, Modulo Arithmetic.
- **Algebra:** Algebraic Expressions, Factorization, Equations, Polynomials, Inequalities, Sequences, Functions.
- **Combinatorics (Counting):** Basic Counting, Pigeonhole Principle, Permutations and Combinations.
- **Geometry:** Measurements (Angles, Lengths, Areas, Volumes), Circles, Triangles, Quadrilaterals, Polygons, Similar Figures, Geometric Transformation.

In this handout, we have included actual questions from IMONST1 so that you can gauge how the difficulty of the questions has varied throughout the years. All credits for IMONST1 problems go to IMO Committee Malaysia. In addition, we have also included problems from other contests of similar style and format to supplement some of the topics covered in this handout.

Abbreviations:
PX - Problem #X from Primary category
JY - Problem #Y from Junior category
SZ - Problem #Z from Senior category

§1 Introductory Problems

Before we begin discussing more advanced topics, consider trying out a few actual problems from IMONST1 that involve logic and **basic counting** techniques.

1. (Sample 1, J18/S13) The positive integers are written in order beginning with 1:

   $$123456789101112131415161718192021\cdots$$

   The 17th digit that gets written is 3 (underlined). What is the 2020th digit that gets written?

2. (Sample 2, J16/S11) Consider the following table:

   $$\begin{array}{ccccccc} 1 & & & & & & \\ 2 & 3 & & & & & \\ 4 & 5 & 6 & & & & \\ 7 & 8 & 9 & 10 & & & \\ 11 & 12 & 13 & 14 & 15 & & \\ \vdots & & & & & & \ddots \end{array}$$

   What is the number directly below 2020?

3. (Sample 3, P8) Hanis draws a circle and two squares on a paper. The figures intersect at several points. What is the maximum possible number of intersection points among these three figures?

4. (2020, P9/J4) Each $\triangle$ symbol in the expression below can be substituted either with $+$ or $-$:

   $$\triangle 1 \triangle 2 \triangle 3 \triangle 4$$

   How many possible values are there for the resulting arithmetic expression? Note: One possible value is -2, which equals $-1-2-3+4$.

5. (2020, J17/S12) A football is made by sewing together some black and white leather patches. The black patches are regular pentagons of the same size. The white patches are regular hexagons of the same size. Each pentagon is bordered by 5 hexagons. Each hexagon is bored by 3 pentagons and 3 hexagons. We need 12 pentagons to make one football. How many hexagons are needed to make one football?

6. (2021, J17/S7) Sofia has forgotten the passcode of her phone. She only remembers that it has four digits and that the product of its digits is 18. How many passcodes satisfy these conditions?

7. (2022, P14/J4) Alya want to complete the following sequence of numbers such that the difference between every pair of adjacent numbers is 1, 3, or 5.

   $$1, \square, \square, \square, \square, \square, 1.$$

   How many ways are there to fill in the blanks with the numbers 2 to 6, each appearing exactly once?

8. (2022, J18/S8) Let n be a positive integer. Vertices of a regular $2n$-gon are painted red and blue in an alternating fashion. All the sides and diagonals of the $2n$-gon are drawn. If the number of segments in the diagram with ends of the same colour is 1122, find the number of segments with ends of different colours.

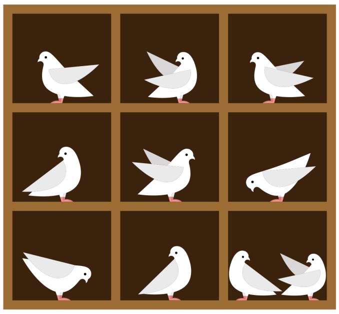
9. (2022, S20) There are 10 cards, consisting of six identical white cards, and one red, blue , green, and yellow card each. On exactly one side of each card is the symbol X. In how many ways is it possible to stack the cards on top of each other such that no two sides of the cards with the symbol X are facing each other?

10. (2023, PB1) We can write 7 as the sum of 3 odd numbers in 2 different ways: $7 = 1 + 1 + 5$ or $7 = 1 + 3 + 3$. How many different ways can we write 13 as the sum of 5 odd numbers?

§2 Pigeonhole Principle

Suppose you are given 9 pigeonholes, and had to put each one of your 10 pigeons into the pigeonholes, is there at least one hole where there are at least two pigeons? (See Figure 1)

Figure 1: Visualisation for $m = 9$



Lemma 2.1 (Pigeonhole Principle)

When $m + 1$ pigeons enter m pigeonholes (m is a positive integer), there must be at least one hole having more than 1 pigeon.

Now consider what happens when you have 29 pigeons in total (instead of just 10) and the number of pigeonholes remains the same as before. What is least number of pigeons we can make sure each pigeonhole does not exceed?

Theorem 2.2 (Generalised Pigeonhole Principle)

When $m+1$ pigeons enter n pigeonholes, there must be one hole having at least $\left\lfloor \frac{m}{n} \right\rfloor + 1$ pigeons.

Let's see what happens when we claim that the answer is 3. This means that in total, the pigeonholes can accommodate at most $9 \times 3 = 27$ pigeons. However, we would have $29 - 27 = 2$ pigeons left. Therefore, we would have no choice but to put each of the 2 pigeons left into a pigeonhole. Thus, the true answer is 4 instead of 3.

As an exercise, try convincing yourself why the following theorem makes sense.

Theorem 2.3 (Infinite Pigeonhole Principle)

When infinitely many elements are partitioned into finitely many sets, there must be at least one set containing infinitely many elements.

[Hint: can you distribute an infinite number of apples into 100 baskets (finite baskets) such that each basket contains a finite number of apples, assuming each basket has unlimited capacity?]

Let's take a look at some examples.

Example 2.4

What is the least number of times I need to flip a coin to get at least 3 heads or 3 tails certainly?

Solution. For this problem, we can use an analogy as follows:

- The possible outcomes of a flip (heads or tails) can be represented by pigeonholes.
- The number of flips can be represented by the pigeons.

You may claim that it is possible to achieve 3 heads or 4 heads using only 4 coin flips. However, you cannot be certain since it is possible that you will just end up getting 2 heads and 2 tails.

What happens if I flipped the coin 5 times? By the Pigeonhole Principle, we will end up having 3 pigeons (flips) in either of the two boxes (possible outcomes).

Hence, the minimum number of coin flips is $\boxed{5}$.

□

Example 2.5 (AHSME 1992)

What is the size of the largest subset S of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

Solution. We want to make sure that for each element within S which is congruent to $x \pmod 7$, there is no other element that is $7 - x \pmod 7$.

By counting the number of elements in $\{1, 2, 3, \ldots, 50\}$ which are $0, 1, \ldots, 5, 6 \pmod 7$ respectively, we note that there are 7 elements congruent to $0, 2, 3, 4, 5, 6 \pmod 7$ but 8 numbers congruent to $1 \pmod 7$.

To maximise the subset, we can put all 8 numbers congruent to $1 \pmod 7$ into S, followed by 7 numbers congruent to $2 \pmod 7$, 7 numbers congruent to $3 \pmod 7$. This choice now eliminates the possibility of putting in any number congruent to $4, 5, 6 \pmod 7$. However, we still have numbers congruent to $0 \pmod 7$. The sum of two numbers congruent to $0 \pmod 7$ is still congruent to $0 \pmod 7$. Hence, we can put at most one of it inside S now. The maximum cardinality of S is $8 + 7 + 7 + 1 = \boxed{23}$. □

Exercise 2.6 (Non-attacking Rooks)

What is the maximum number of rooks that can be placed on an 8×8 chessboard such that each row and column contains no more than 1 rook?

Exercise 2.7 (AMC 10B 2005)

A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?

§3 Principle of Inclusion-Exclusion

§3.1 Dealing with Overcounting

PIE is essentially a special application of one of the most fundamental counting techniques: strategic overcounting. In a typical PIE computation, we will repeatedly overcount and undercount until, at the end of the process, we arrive at exactly the correct count. To illustrate the counting process in PIE, let's consider the simplest example:

Example 3.1 (A Two-set Problem)
There are 15 students taking Japanese and 12 taking French. Of these, 7 are taking both Japanese and French. How many students are in at least one of the classes?

Solution. It is easy to think that the answer is $15 + 12 = 27$. However, here's the catch: some students are in both classes and therefore are counted twice in the sum $15 + 12$. How do we correct for counting some students twice?

Since we know how many students are counted twice in $15 + 12$, we can subtract them once to have them only be counted once. Thus, our desired answer is

$$15 + 12 - 7 = \boxed{20}$$

□

We can actually write the idea above out in mathematical terms as follows:

- Let A be the set of students taking Japanese.
- Let B be the set of students taking French.

This gives us the result below.

Lemma 3.2 (Two Sets)
Given two sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Exercise 3.3 (A Larger Universal Set)
There are 45 students in a class. 4 of them don't take any language subjects, 8 takes German **only**, 10 takes both German and Chinese and the rest takes Chinese **only**. How many students take Chinese as a subject?

§3.2 Attempting to Generalise

Can we generalise this fact? The next sensible step to take is to consider what happens when we have 3 sets and above.

Lemma 3.4 (Three Sets)
Given three sets A, B and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Example 3.5

How many positive integers less than 450 are relatively prime to 450?

Solution. We can first alter the problem by considering the number of positive integers **less than or equal to** 450 coprime to 450, rather than less than 450. The reason for this is because 450 isn't coprime to itself, hence it does not affect our final answer.

You might have realised that there's no obvious way to count the answer directly, so we have to try complementary counting.

Since $450 = 2 \times 3^2 \times 5^2$, we know that a positive integer is not coprime to 450 if it is divisible by at least one of the prime factors $2, 3$ or 5. The number, N of positive integers that satisfy this condition is

$$|\{x|x \text{ is divisible by 2}\}| + |\{x|x \text{ is divisible by 3}\}| + |\{x|x \text{ is divisible by 5}\}|$$
$$- |\{x|x \text{ is divisible by 2 and 3}\}| - |\{x|x \text{ is divisible by 3 and 5}\}| - |\{x|x \text{ is divisible by 5 and 2}\}|$$
$$+ |\{x|x \text{ is divisible by 2, 3 and 5}\}|$$

Therefore,

$$\begin{aligned} N &= \frac{450}{2} + \frac{450}{3} + \frac{450}{5} - \frac{450}{6} - \frac{450}{10} - \frac{450}{15} + \frac{450}{30} \\ &= 225 + 150 + 90 - 75 - 45 - 30 + 15 \\ &= 330 \end{aligned}$$

However, our desired answer is $450 - N = 450 - 330 = \boxed{120}$. □

Remark 3.6. This problem has a huge significance in number theory in that there is an actual function dedicated to obtaining the answer to the problem above "directly".

Euler's totient function, $\phi(n)$ counts the number of positive integers up to a given integer n that are relatively prime to n. A well-known formula to obtain this value is

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, \ldots, p_k$ are distinct prime factors of n. By using this formula, the answer to our question earlier is

$$\phi(450) = 450\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = \boxed{120}.$$

Theorem 3.7 (Inclusion-Exclusion Principle)

Given finite sets $A_1, A_2, \ldots, A_n$. Then,

$$\left|\bigcup_{1\le i\le n} A_k\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}\left|\bigcap_{1\le i\le n} A_i\right|$$

Exercise for the reader: Prove the result above by induction!

You might have noticed that PIE is also used when calculating probabilities for non-mutually exclusive events. In fact, we can replace "cardinality" with "probability", which gives us a new formula:

Corollary 3.8 (Inclusion-Exclusion Principle for Probabilities)

Given finite sets $A_1, A_2, \ldots, A_n$. Then,

$$\left|\bigcup_{1\leq i\leq n} A_k\right| = \sum_{i=1}^{n}\mathbb{P}(A_i) - \sum_{1\leq i<j\leq n}\mathbb{P}(A_i\cap A_j) + \sum_{1\leq i<j<k\leq n}\mathbb{P}(A_i\cap A_j\cap A_k)$$
$$-\cdots+(-1)^{n+1}\mathbb{P}\left(\bigcap_{1\leq i\leq n} A_i\right)$$

Remark 3.9. Although it seems convenient, you should not memorise a "formula" for PIE. Instead, think about how many times each item is counted, and make sure that each item is counted once and only once.

§3.3 Advanced Inclusion-Exclusion

We've seen that PIE is quite handy when it comes to counting items that have "at least 1" set of properties. What if we want to count items that have "at least 2" of a set of properties?

Example 3.10

Five standard 6-sided dice are rolled. What is the probability that at least 3 of them show a six?

Solution. We start by counting the number of favorable outcomes where at least 3 dice show a six.

1. Three sixes: We choose 3 out of the 5 dice to show sixes, which can be done in $\binom{5}{3}$ ways. The remaining 2 dice can show any number from 1 to 6, giving us 6^2 possible outcomes for these dice. Thus, the number of outcomes with exactly 3 sixes is:

$$\text{Outcomes with 3 sixes} = \binom{5}{3}\cdot 6^2 = 10\cdot 36 = 360$$

2. Four sixes: We have overcounted the cases where 4 dice show sixes in the previous step. For each subset of 4 dice showing sixes (which can occur in $\binom{5}{4}$ ways), we have counted one subset of 3 dice showing sixes multiple times (specifically, 3 times). Thus, we subtract:

$$\text{Correction for 4 sixes} = 3\cdot\binom{5}{4}\cdot 6 = 3\cdot 5\cdot 6 = 90$$

3. Five sixes: Finally, we need to account for the case where all 5 dice show sixes. This case was counted $+10$ times in the first step (since $\binom{5}{3} = 10$) and -15 times in the second step (since $\binom{5}{4} = 5$). The net count for this case is -5, so we need to add back 6 to correctly count this case:

$$\text{Correction for 5 sixes} = 6 \cdot \binom{5}{5} = 6 \cdot 1 = 6$$

Adding these together gives us the total number of successful outcomes:

$$\text{Total successful outcomes} = 360 - 90 + 6 = 276$$

The final expression summarizing the computation is:

$$\binom{5}{3} \cdot 6^2 - 3 \cdot \binom{5}{4} \cdot 6 + 6 \cdot \binom{5}{5} = 276$$

Therefore, the desired probability is

$$\mathbb{P}(\text{5 dice show at least 3 sixes}) = \frac{276}{6^5} = \boxed{\frac{23}{648}}$$

□

Remark 3.11. An easier solution to the problem above is the consider the binomial distribution. Let the random variable X be the number of sixes obtained from rolling five dice at a time. Since $X \sim \text{Bin}(5, 1/6)$,

$$\begin{aligned}\mathbb{P}(X \geq 3) &= \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5) \\ &= \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) + \binom{5}{5}\left(\frac{1}{6}\right)^5 \\ &= \boxed{\frac{23}{648}},\end{aligned}$$

which is the same as what we had obtained earlier.

§4 Permutations and Combinations

§4.1 Sampling and distribution

Problems that involve the understanding of permutations and combinations can usually be divided into two categories:

- Sampling: How many distinct samples of size r can be drawn from the n objects?
- Distributions: How many ways are there to distribute r balls into n distinguishable urns?

Before we begin solving these problems, it helps to ask the following questions: For sampling,

1. Are the samples taken with or without replacement (that is, can we pick an object at most once, or can we pick the same object more than once)?
2. Does the order in which we select the items in the sample matter?

For distributions,

1. Can an urn hold at most one ball (exclusive) or can it hold many balls (non-exclusive)?
2. Are the balls distinguishable (distinct) or are they indistinguishable (identical)?

Example 4.1 (Sampling)

Here are four different cases to consider:

1. A club consisting of 26 members needs to select an executive board consisting of a president, a vice-president, a secretary and a treasurer. Each position has different duties and responsibilities. No individual can hold more than one office.
2. A club consisting of 26 members needs to select a delegation of 4 (distinct) members to attend a convention. The delegation of 4 wears identical goofy hats.
3. A club consisting of 26 members requires volunteers to complete 4 distinct chores; sweeping the clubhouse printing off raffle tickets for the drawing at the party that night, picking up the prizes and picking up the empties after the party. The same person can volunteer for more than one job, and each job requires only volunteer.

Solution.

	Order matters?	**With replacement?**
Question 1	Yes	No
Question 2	No	No
Question 3	Yes	Yes

The information in the table above tells us how the answer to each question in computed differently.

1. Answer: $26 \times 25 \times 24 \times 23 =^{26} P_4 = 358000$
2. Answer: $\binom{26}{4} = \frac{1}{4!}{}^{26}P_4 = 14950$
3. Answer: $26^4 = 456976$

□

Example 4.2 (Distribution)

Similarly, we consider four different cases for distribution as well:

1. In how many ways can 4 executive board positions (distinguishable balls) be distributed among 26 members (urns) with exclusion (since no member can hold more than one position)?
2. In how many ways can 4 delegation slots (indistinguishable balls) be distributed among 26 members (urns) with exclusion?
3. In how many ways can 4 different jobs (distinguishable balls) be distributed among 26 members (urns) without exclusion (since one member can do multiple jobs)?

Solution.

	Exclusive?	**Distinguishable?**
Question 1	Yes	No
Question 2	No	No
Question 3	Yes	Yes

The information in the table above tells us how the answer to each question in computed differently.

The answers to Questions 1 to 3 are the same as their respective sampling analogue in the previous example.

1. Answer: $26 \times 25 \times 24 \times 23 =^{26} P_4 = 358000$
2. Answer: $\binom{26}{4} = \frac{1}{4!}{}^{26}P_4 = 14950$
3. Answer: $26^4 = 456976$

□

§4.2 Stars and bars problem

Example 4.3

How many ways are there to arrange 7 identical stars and 4 identical bars?

Solution. There are $11!$ ways to arrange 11 distinct objects. Since the stars are all identical, we must eliminate permutations of stars by dividing with $7!$. Similarly, we eliminate permutations of bars by dividing with $4!$. Hence, there are

$$\frac{11!}{7!4!} = \binom{11}{4} = \boxed{330}$$

ways to arrange them. □

Example 4.4

How many distinct 5-tuples of nonnegative integers $(x_1, x_2, x_3, x_4, x_5)$ satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 7?$$

Solution. Note how this problem is analogous to the previous one.

- The sum to be achieved is the number of stars.
- The number of terms on the LHS is the number of *bins* between bars.

$$\star\ \star\ \star\ \star\ |\ |\ \star\ |\ \star\ \star\ |$$

Figure 2: These four bars give rise to five bins containing 4, 0, 1, 2, and 0 objects

To visualise bins, think of them as spaces where the stars can be placed - before the 1st bar, between any bars, or after the 4th bar. This means that there are 5 bins (terms), which corresponds to $5 - 1 = 4$ bars. Hence, the answer to this problem is also $\binom{11}{4} = \boxed{330}$ □

This allows us to invoke another analogy.

Example 4.5
How many ways are there to distribute 7 indistinguishable balls into 5 indistinguishable urns?

Solution. 7 balls and 5 urns are equivalent to 7 stars and 5 bins, so the answer is

$$\binom{7+5-1}{5-1} = 330$$

□

Let's put this concept in a formal description.

Theorem 4.6
For any positive integers n and k, the number of k-tuples of nonnegative integers, whose sum is n, is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Exercise 4.7
How many distinct 3-tuples of positive integers (x_1, x_2, x_3) satisfy the equation

$$x_1 + x_2 + x_3 = 10?$$

Exercise 4.8 (IMONST 2020)
How many 3-digit number have its sum of digits equal to 4?

Exercise 4.9
How many 5-digit positive integers $\overline{x_1x_2x_3x_4x_5}$ are there such that $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$?

§4.3 A Summary

The following diagram shows the complete set of correspondences between sampling and distribution. Although the formulation of a sampling problem may appear to be quite dissimilar to the corresponding distribution problem, the two are in fact mathematically equivalent.

Samples of size r from n distinguishable objects	**Without replacement**	**With replacement**	
Order matters	${}^nP_r$	n^r	**Distinguishable balls**
Order doesn't matter	$\binom{n}{r}$	$\binom{n+r-1}{r}$	**Indistinguishable balls**
	Exclusive	**Non-exclusive**	**Distributions of r balls into n distinguishable urns**

Exercise 4.10

1. How many ways are there to distribute 6 distinguishable balls into 3 distinguishable urns?
2. How many ways are there to distribute 6 distinguishable balls into 3 indistinguishable urns?
3. How many ways are there to distribute 6 indistinguishable balls into 3 distinguishable urns?
4. How many ways are there to distribute 6 indistinguishable balls into 3 indistinguishable urns?

§5 Bonus Problems

The following questions are sorted chronologically (not by difficulty).
Permutations & Combinations

1. (Sample 1, J11/S6) Determine the number of even integers between 4000 and 9000 with all four digits different.

2. (Sample 2, P13/J8/S3) Given a six-digit number N. How many seven-digit numbers are there such that when we remove one of its digits, the result is N?

3. (Sample 2, P15/J10/S5) A math club has 7 members. The club advisor wanted to choose at least 4 members to represent the club in a math contest.

4. (Sample 2, J20/S15) There are 99 students (call them Student 1, Student 2, and so on) and 99 seats around a round table. How many ways are there to arrange the seating of all the students so that for any two neighbouring students, their numbers differ by at most 2?

5. (2021, S19) A company has a secrete safe box that is locked by six locks. Several copies of the keys are distributed among the directors of the company. Each key can unlock exactly one lock. Each director has three keys for three different locks. No two directors can unlock the same three locks. No two directors together can unlock the safe. What is the maximum possible number of directors in the company?

6. (2022, P3) How many ways are there to arrange the letters in IMONST such that the vowels, I and O appear in alphabetical order, and the consonants M, N, S and T also appear in alphabetical order? For example, IMONST and MNISTO are valid arrangements but MONIST is not.

7. (2022, P12) Maria has 5 blocks of weight 1,2,3,4 and 5 kg respectively. She separate the blocks into two boxes in such a way that the weight of the first box is greater than the second box and that there are more blocks in the first box than in the second. In how many different ways can she do this?

8. (2022, P13) My school's chess club invited another school's chess club to practice together. Each player from my school played against 4 different players from the other school, while each player from the other school played against 3 different players from my school. If my school had 12 players, how many players did the other school have?

9. (2022, J15) How many positive integers of 10 digits $\overline{abcdefghij}$, all of which are different, are there such that $a+j=b+i=c+h=d+g=e+f=9$?

10. (2022, S11) In how many ways can we arrange the letters in IMONST so that the consecutive letters in the alphabet are not next to each other? For example, MINSOT and ISMOTN are valid arrangements but IMONST and TIMONS are not.

11. (2023, PC1) Determine the number of different arrangements of the letters in the word $GAUSS$.

12. (2023, PC2) How many years in the 21st century consist of four different digits?

13. (2023, JC7) We list down, in increasing order, all five-digit numbers that can be written using the digits $1, 2, 3, 4, 5$. What is the 88th number in the list?

14. (2023, SB6) Two boys and two girls are arranged in a straight line at random. The probability that thee two girls are between the two boys is $1/k$. What is k?

15. (2023, SC6) How many four-digit number are there such that at least one digit appears more than once?

16. Xander forms a 6-digit passcode for his locker using only the digits $1, 2$ and 3. In addition, each distinct digit is used exactly twice and no identical digit are adjacent to each other. In the worst-case scenario, how many tries does it take for one to unlock his locker given the information above?

17. How many five-digit numbers are there that have a digit sum of 13?

Pigeonhole Principle

19. (2022, J11) Rahman knows that his sock drawer contains 12 black socks, 10 blue socks, 8 gray socks and 6 brown socks, all of the same shape and all mixed together. He is about to go on a trip and wants to take at least two pairs of matching socks of any two different colours. Since it is dark and he cannot tell the colours apart, he randomly grabs a bunch of socks. How many socks will he need to pack, at least, to be sure that he has two matching pairs of two different colours?

Venn Diagrams and Inclusion-Exclusion

20. (Sample 1, J4) Among 250 students at a school, 150 have taken part in the Mathematics Olympiad and 130 in the Science Olympiad. Each student participates in at least one Olympiad. How many students have participated in both Olympiads?

21. (2022, P6) In a lake, there are two kinds of fish: red and yellow .Of all the fish in the lake, two fifths are yellow, while the others are red. Three quarters of all the yellow fish are female. If the total number of female fish equals the total number of male fish, what is the percentage of red male fish in the lake?

22. (2023, SB10) Among 100 students in a class, 93 students know French, 73 students know Japanese, 69 students know Arabic, and 65 students Swahili. What is the least possible number of students who know all four languages?

Recursion

23. (Sample 3, P20/J15/S10) Given a sequence of integers

$$a_1, a_2, a_3, \dots,$$

Starting from the third term, each term is equal to the sum of all previous terms (for example, $a_5 = a_1 + a_2 + a_3 + a_4$). It is know that $a_1 = 1$, and n is the largest possible integer with $a_n = 10000$. Find a_2.

24. (2021, S20) The cells of a 2021×2021 table are filled with numbers using the following rule. The bottom left cell, which we label with coordinate $(1, 1)$, contains the number 0. For every other cell C, we consider a route from $(1, 1)$ to C, where at each step we can only go one cell to the right or one cell up (not diagonally). If we take the number of steps in the route and add the numbers from the cells along the route, we obtain the number in cell C. For example, the cell with coordinate $(2, 1)$ contains $1 = 1 + 0$, the cell with coordinate $(3, 1)$ contains $3 = 2 + 0 + 1$, and the cell with coordinate $(3, 2)$ contains $7 + 3 + 0 + 1 + 3$. What is the last digit of the number in the cell $(2021, 2021)$?

25. (2023, SC8) We are given 10 line segments, each with an integer length. No three segments can form a triangle. What is the minimum length of the longest line segment?

Miscellaneous

26. (2020, S19) A set S has 7 elements. Several 3-element subsets of S are listed, such that any 2 listed subsets have exactly 1 common element. What is the maximum number of subsets that can be listed?

27. (2021, P18) Hafiz marks k points on the circumference of a circle. he connects every point to every other point with straight lines. If there are 210 lines formed, what is k?

28. (2022, P17) We want to choose 3 positive integers a, b and c satisfying $a < b < c < 2022$ and $b = (a+c)/2$. How many ways are there to do it?

29. (2022, J20) Consider the following shape consisting of five triangles all meeting at a common vertex in the middle. Starting at the middle, how many paths can we use to draw this shape with a pen without lifting the pen and without repeating any edge?